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An Improved Derivation of the LSZ Asymptotic Condition within the Framework of the Haag-Ruelle Scattering Theory

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Abstract

It is pointed out that Hepp's derivation of the LSZ asymptotic condition is not quite correct. Rigorous proof is given for an even wider class of asymptotic states by direct use of Haag's method. This is possible because of two lemmas, proved in the appendix, one generalising Ruelle's estimate on smooth Klein-Gordon wave functions and the other stating that every tempered test function can be represented as a product of two other tempered test functions.

1. *Introduction*

K. Hepp is generally believed to have given a rigorous derivation of the LSZ asymptotic condition within the framework of the (restricted) Haag-Ruelle scattering theory (Hepp, 1965) (see also Hepp, 1966). Actually, there is an error in his proof of rapid strong convergence (in time) of Haag's almost localised states for non-overlapping *tempered* test functions (see Hepp (1965), proof of Theorem 2.1).

His argument is based on the *assumption* that for every $\chi \in \mathscr{S}(R^{3(k-1)})$, $k \geqslant 3$, with

$$\left[m^2 + \left(\sum_{j=2}^{k} \mathrm{p}_j\right)^2\right]^{-1/2} \sum_{j=2}^{k} \mathrm{p}_j + (m^2 + \mathrm{p}_2^2)^{-1/2}\mathrm{p}_2 \neq 0 \qquad \text{in supp}\, \chi$$

there exists a suitable $\mathcal{O}_{\mathcal{M}}$-partition $\{\alpha_l\}$ $(1 \leqslant l \leqslant 3)$ of the unity for which

$$\left[m^2 + \left(\sum_{j=2}^{k} \mathbf{p}_j\right)^2\right]^{-1/2} \sum_{j=2}^{k} p_j^l + (m^2 + \mathbf{p}_2^2)^{-1/2} p_2^l \neq 0 \qquad \text{in supp } \chi\alpha_1$$

However, as may be easily seen (Appendix 3), those partitions do not exist in general. Thus Hepp's proof is only justifiable if his definition of *non-overlapping* test functions is sharpened to mean non-zero minimal distance of supports in momentum space†—then, indeed, existence of the required partition of unity is generally guaranteed.

Of course, a restriction of this kind does not seem to be crucial from a physical point of view. On the contrary, according to physical intuition it should be possible, without affecting the results, to enlarge Hepp's class of non-overlapping test functions to the class of 'essentially non-overlapping' test functions:

Definition. A set $\{\hat{f}_j\} \subset \mathcal{S}(R^3)$ is called *essentially non-overlapping* if supp $\hat{f}_j \cap$ supp $\hat{f}_l$ has no interior points for $j \neq l$.

Now, the purpose of the present paper is to give rigorous proof of the LSZ asymptotic condition in Hepp's form for *essentially* non-overlapping asymptotic states. We will not try to cure Hepp's proof but rather apply the original Haag method (Haag, 1958) as extended in Araki & Haag (1967), which is physically more transparent and well suited for our purposes.

2. *Assumptions and Corresponding Results of the Haag–Ruelle Theory*

In order to keep things as simple as possible, without thereby circumventing principal difficulties, however, let us restrict ourselves to the theory of one kind of neutral scalar particle with mass $m > 0$, described by the self-interacting tempered Wightman field $A(x)$.

This means, for all $\varphi \in \mathcal{S}(R^4)$ the formal integrals $\int dx\, A(x)\varphi(x)$ (also denoted by $A(\varphi)$) represent linear operators, well defined on a common invariant dense linear manifold D in a separable Hilbert space $\mathcal{H}$. According to *Einstein causality* these operators are to commute on D for φ's with spacelike separated supports, i.e. symbolically:‡

$$[A(x), A(y)] = 0 \qquad \text{for } (x-y)^2 < 0$$

Furthermore, a strongly continuous unitary representation $U(\Lambda, a)$ of the connected Poincaré group is defined on $\mathcal{H}$, for which we have

$$U(\Lambda, a)A(x)U^{-1}(\Lambda, a) = A(\Lambda x + a), \qquad U(\Lambda, a)D \subset D.$$

There is a *vacuum* state vector Ω in D, unique up to a phase factor, which is cyclic for the smeared fields $A(\varphi)$ and invariant under the representation $U(\Lambda, a)$. Finally, the *spectrum condition* requires the spectrum of the energy-momentum operator P^μ (its components being the generators of the represen-

† Since we shall restrict ourselves to particles of only one kind it is not necessary to refer to velocity space.

‡ We use the notation $px = p^0x^0 - \mathbf{px}$ for the Lorentz scalar product.

tation $U(1, a)$ of the translation group) to lie in the closed foreward lightcone $\overline{V}_+$.

In addition to these so-called *Wightman axioms* (Wightman, 1956; Streater & Wightman, 1964) we have to impose two further conditions in order to make use of the Haag-Ruelle theory (Haag, 1958; Ruelle, 1962). First, we have to postulate the existence of a *mass gap* in the spectrum of P^μ, i.e. the spectrum of P^μ is to be contained in $\overline{V_+^m} \equiv \{p : p \in V_+, p^2 \geqslant m^2\}$, except for the eigenvalue 0 corresponding to the vacuum. Secondly, we have to postulate the existence of almost local fields creating 1-particle states from the vacuum. Following Hepp, we even require the 1-particle states to be created from the vacuum by the smeared fields $A(\varphi)$ themselves in order to allow for derivation of dispersion relations (Hepp, 1964). Explicitly, we assume the following structure of the 2-point function:

$$\langle 0 \,|\, A(x)A(y) \,|\, 0 \rangle = i\Delta_m^+(x-y) + i\int_M^\infty d\rho(\mu)\Delta_\mu^+(x-y) \tag{2.1}$$

where $M > m$.

Now, in Hepp's notation (1964) for $\varphi \in \mathscr{S}(R^4)$

$$\tilde{\varphi}(p) \equiv (2\pi)^{-2} \int dx \varphi(x)\, e^{ipx}$$

$$w_{\mathbf{p}} \equiv (\mathbf{p}^2 + m^2)^{1/2}$$

$$\varphi(x, t) \equiv (2\pi)^{-5/2} \int dp \varphi(p) \left(\frac{p^0 + w_{\mathbf{p}}}{2w_{\mathbf{p}}} \right) \exp[i(p^0 - w_{\mathbf{p}})t] \, \exp(-ipx)$$

$$A(\varphi, t) \equiv \int dx A(x) \varphi^*(x, t)$$

the following theorem is a special result of the Haag-Ruelle theory (Haag, 1958; Ruelle, 1962):

Theorem 1. *Let* $\varphi_1, \ldots, \varphi_n$ *be elements of* $\mathscr{S}(R^4)$. *If* supp $\tilde{\varphi}_j \subset \{p : p^2 < M^2\}$ *then*

$$\underset{t \to \pm\infty}{\text{s-lim}} \prod_{j=1}^n A^*(\varphi_j, t)\Omega$$

exists and represents an asymptotic state corresponding to n particles with momentum space wave functions $\hat{f}_j(\mathbf{p}) \equiv \tilde{\varphi}_j(w_{\mathbf{p}}, \mathbf{p})$.

To be sure, within the Haag-Ruelle frame this theorem is 'only' proved for φ_j's with Fourier transforms of the form $\tilde{\varphi}_j(p) = \tilde{g}_j(\mathbf{p})\tilde{h}_j(p)$ with $\tilde{g}_j \in \mathscr{S}(R^3)$ and $\tilde{h}_j \subset \mathscr{S}(R^4)$ (compare proof of Theorem 2, below). However, contrary to Hepp's opinion (Hepp, 1964), this is no real restriction:

Lemma 1. *Let n, n' be positive integers with* $n' \leqslant n$ *and let* $\varphi_0, \varphi_1, \varphi_2, \ldots$ *be a (countably infinite) sequence of test functions in* $\mathscr{S}(R^n)$. *Then there is a positive* $g \in \mathscr{S}(R^{n'})$ *and a sequence* $\{h_0, h_1, h_3, \ldots\} \subset \mathscr{S}(R^n)$ *such that*

$$\varphi_r(t^1, \ldots, t^n) = g(t^1, \ldots, t^{n'})h_r(t^1, \ldots, t^n) \qquad \text{for } r = 0, 1, 2, \ldots$$

Proof. See Appendix 1.

3. *Proof of the* LSZ *Asymptotic Condition*

The crucial point in the derivation of the LSZ asymptotic condition is well known to be the rapid strong convergence of Haag's almost localised states

$$\prod_{j=1}^{n} A^*(\varphi_j, t)\Omega$$

for essentially non-overlapping $\{\tilde{\varphi}_j(w_{\mathbf{p}}, \mathbf{p})\} \subset \mathscr{S}(R^3)$ in the limit $t \to \pm\infty$.

In order to prove this convergence we profitably exhibit the asymptotic properties of smooth Klein-Gordon wave functions

$$f(x) = (2\pi)^{-3/2} \int \frac{d\mathbf{p}}{2w_{\mathbf{p}}} \hat{f}(\mathbf{p}) \exp[-i(w_{\mathbf{p}}x^0 - \mathbf{p}\mathbf{x})]; \qquad \hat{f}(\mathbf{p}) \in \mathscr{S}(R^3) \tag{3.1}$$

as outlined by Haag (1958), Araki & Haag (1967). Then the following generalisation of the original Ruelle lemma (Ruelle, 1962) just states what one is to expect from heuristic physical considerations:†

Lemma 2. *Let N be a non-negative integer. Then there are constants, A, B, C for which the following statements hold:*

1. $\|(t, \mathbf{v}t)\|^N |f(t - x^0, \mathbf{v}t - \mathbf{x})| \leqslant A(1 + \|x\|^N) \max_{|\mathbf{a}| < B} \max_{\mathbf{p} \in R^3} |(1 + \|\mathbf{p}\|^C)$

   $\times D_{\mathbf{p}}^{\mathbf{a}} \hat{f}(\mathbf{p})|$ *for arbitrary* $\hat{f} \in \mathscr{S}(R^3)$, $t \in R^1, x \in R^4$, *and*

   $\mathbf{v} \in R^3 - \left\{\frac{\mathbf{p}}{w_{\mathbf{p}}} : \mathbf{p} \in \operatorname{supp} \hat{f}\right\}.$

2. $|t|^{3/2} |f(t, \mathbf{v}t)| \leqslant A \max_{|\mathbf{a}| < B} \max_{\mathbf{p} \in R^3} |(1 + \|\mathbf{p}\|^C) D_{\mathbf{p}}^{\mathbf{a}} \hat{f}(\mathbf{p})|$ *for arbitrary*

   $\hat{f} \in \mathscr{S}(R^3), t \in R^1, \mathbf{v} \in R^3.$

(*f defined by* (3.1))

† The $\hat{f}$-dependence of the majorisation constants (see also Lemma 3, below) will not be exhibited in the present paper. See the first of our concluding remarks, however. We adopt the usual notation

$$\|t\| = \left(\sum_{j=1}^{n} (t^j)^2\right)^{1/2}, \qquad t^a = \prod_{j=1}^{n} (t^j)^{a^j}, \qquad |a| = \sum_{j=1}^{n} a^j$$

$$D_t^a = \frac{\partial^{|a|}}{\partial_{t^1}^{a^1} \partial_{t^2}^{a^2} \ldots \partial_{t^n}^{a^n}}$$

for $t = (t^1, \ldots, t^n) \in R^n$ and $a \in Z_+^n$ (Z_+^n being the set of n-vectors with non-negative integer components).

Proof. While statement 2 was proved by Araki (1962), the proof of statement 1 is given in Appendix 2.

Statement 1 of this lemma shows that particles in essentially non-overlapping states (essentially non-overlapping momentum space wave functions $\hat{f}_j$) become strongly separated for large times:

Lemma 3. *Let N be a non-negative integer. Then there are constants A, B, C such that the inequality*

$$|t^N f_1(t, \mathbf{x}_1) f_2(t, \mathbf{x}_2)| \leqslant A \max_{j\in\{1,2\}} \max_{|\mathbf{a}|<B} \max_{\mathbf{p}\in R^3} |(1+\|\mathbf{p}\|^C) D_{\mathbf{p}}^{\mathbf{a}} \hat{f}_j(\mathbf{p})|$$

holds for arbitrary essentially non-overlapping $\{\hat{f}_1, \hat{f}_2\} \subset \mathscr{S}(R^3)$, $t \in R^1$, $\{\mathbf{x}_1, \mathbf{x}_2\} \subset R^3$ *with* $(\mathbf{x}_1 - \mathbf{x}_2)^2 < |t|$.

(f_1, f_2 *defined according to* (3.1).)

Proof. By statement 1 of Lemma 2 the inequality

$$|x^0|^N |f_j(x)| \leqslant A \max_{|\mathbf{a}|<B} \max_{\mathbf{p}\in R^3} |(1+\|\mathbf{p}\|^C) D_{\mathbf{p}}^{\mathbf{a}} \hat{f}_j(\mathbf{p})|$$

holds for $\hat{f}_i \in \mathscr{S}(R^3)$ within the region

$$K_{\hat{f}_j} \equiv \left\{ x = (t, \mathbf{v}t + \mathbf{y}) : t \in R^1, \mathbf{v} \in R^3 - \left\{ \frac{\mathbf{p}}{w_{\mathbf{p}}} : \mathbf{p} \in \operatorname{supp} \hat{f}_j \right\}, \mathbf{y} \in R^3, \mathbf{y}^2 < |t| \right\}$$

with suitable constants A, B, C independent of $\hat{f}_j$ ($j = 1, 2$). Hence, the statement of Lemma 3 is an immediate consequence of the fact that $(t, \mathbf{x}_j) \in R^4 - K_{\hat{f}_j}$ ($j = 1, 2$) implies $(\mathbf{x}_1 - \mathbf{x}_2)^2 > |t|$ for essentially non-overlapping $\{\hat{f}_1, \hat{f}_2\}$.

So we are ready to prove the required rapid strong convergence of appropriate states:

Theorem 2. *Let* $\{\hat{f}_1, \ldots, \hat{f}_n\} \subset \mathscr{S}(R^3)$ *be essentially non-overlapping. Then there are* $\varphi_1, \ldots, \varphi_n \in \mathscr{S}(R^4)$ *with* $\tilde{\varphi}_j(w_{\mathbf{p}}, \mathbf{p}) = \hat{f}_j(\mathbf{p})$ *for* $j \in \{1, \ldots, n\}$ *and positive constants* $C_0, C_1, \ldots$ *such that*

$$\max_{N\in Z_+} \max_{t\in R^1} C_N |t|^N \left\| \frac{d}{dt} \prod_{j=1}^{n} A^*(\varphi_j, t)\Omega \right\| < A$$

Proof. By Lemma 1 there is a positive valued function $\hat{g} \in \mathscr{S}(R^3)$ such that $\hat{f}_j' \equiv \hat{f}_j / \hat{g} \in \mathscr{S}(R^3)$ for $j \in \{1, \ldots, n\}$. Let us choose some $h \in \mathscr{S}(R^1)$ with $\operatorname{supp} h \subset (0, M^2)$, $h(m^2) = 1$ and define:

$$\tilde{\varphi}(p) \equiv \hat{g}(\mathbf{p}) h(p^2), \qquad \tilde{\varphi}_j(p) \equiv \hat{f}_j'(\mathbf{p}) \tilde{\varphi}(p).$$

So by use of the almost localised field

$$B(x) \equiv (2\pi)^{-2} \int dx' A(x') \varphi(x' - x)$$

and the smooth positive frequency Klein-Gordon wave functions

$$f_j'(x) \equiv (2\pi)^{-3/2} \int \frac{d\mathbf{p}}{2w_{\mathbf{p}}} \hat{f}_j'(\mathbf{p}) \exp[-i(w_{\mathbf{p}} x^0 - \mathbf{p}\mathbf{x})]$$

we may write

$$A^*(\varphi_j, t) = i \int_{x^0 = t} d\mathbf{x} \left(B(x) \frac{\partial f_j'(x)}{\partial x^0} - \frac{\partial B(x)}{\partial x^0} f_j'(x) \right)$$

for $j = 1, \ldots, n$. Now we just have to copy one of Ruelle's proofs (see Ruelle, 1962, p. 158):

Expand

$$\left\| \frac{d}{dt} \prod_{j=1}^{n} A^*(\varphi_j, t)\Omega \right\|^2$$

into its *finite* sum of products of truncated vacuum expectation values which are of the form

$$I(t) = \int d\mathbf{x}_0\, d\mathbf{x}_1 \ldots d\mathbf{x}_k f_0''(\mathbf{x}_0, t) f_1''(\mathbf{x}_1, t) \ldots f_k''(\mathbf{x}_k, t)$$
$$\times \tilde{F}'(\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_k - \mathbf{x}_{k-1})$$

with $\tilde{F}' \in \mathscr{S}(R^{3k})$ and smooth (positive frequency) Klein-Gordon wave functions f_0'' (resp. $f_0''^*$), ..., f_k'' (resp. $f_k''^*$). Products containing only factors $I(t)$ with $k < 2$ do not contribute since (2.1) implies $A(\varphi_j, t)\Omega = (d/dt) \times A^*(\varphi_j, t)\Omega = 0$. From Lemma 2 we see that all the truncated vacuum expectation values $I(t)$ are bounded in t. Moreover, for $k \geqslant 2$ there are at least two of the $f_0''^{(*)}, \ldots, f_k''^{(*)}$ which form an essentially non-overlapping set in momentum space. Therefore, by Lemmas 2 and 3, we see $|t^N I(t)|$ to be bounded in t for every non-negative integer N in this case. So all the products vanish sufficiently rapidly for $t \to \pm\infty$.

The few remaining steps in the derivation of the LSZ asymptotic condition may be literally taken over from Hepp (1965) to give the final result:

Theorem 3. *Let* $\varphi \in \mathscr{S}(R^4)$ *and let* $\{\varphi_1, \ldots, \varphi_n\} \subset \mathscr{S}(R^4)$ *correspond to essentially non-overlapping* $\{\hat{f}_1, \ldots, \hat{f}_n\} \subset \mathscr{S}(R^3)$ $(\hat{f}_j(\mathbf{p}) \equiv \tilde{\varphi}_j(w_\mathbf{p}, \mathbf{p}))$. *If* $\operatorname{supp} \tilde{\varphi}_{(j)} \subset \{p\colon p^2 < M^2\}$ *we have*

$$\operatorname*{s-lim}_{t \to \pm\infty} A^*(\varphi, t) \operatorname*{s-lim}_{t' \to \pm\infty} \prod_{j=1}^{n} A^*(\varphi_j, t')\Omega = \operatorname*{s-lim}_{t \to \pm\infty} A^*(\varphi, t) \prod_{j=1}^{n} A^*(\varphi_j, t)\Omega$$

(*the interpretation being given by Theorem* 1).

Let us conclude our considerations by three remarks:

1. By use of the explicit $\hat{f}$-dependence of the majorisation constants in Lemmas 2 and 3 Theorem 3 may be easily generalised to the strong $t \to \pm\infty$ limit of $A^*(\varphi, t)$ applied to asymptotic states with arbitrary n-particle momentum space wave functions from the *complete* linear subspace of $\mathscr{S}(R^{3n})$ spanned by *all* functions of the form $\hat{f}_1(\mathbf{p}_1)\hat{f}_2(\mathbf{p}_2) \ldots \hat{f}_n(\mathbf{p}_n)$ with essentially non-overlapping $\{\hat{f}_j\} \subset \mathscr{S}(R^3)$.
2. The methods used here are well adapted to *explicit* generalisation to Jaffe fields (Jaffe, 1967) (see also Lücke, 1973).

3. It should be clear how to apply Lemmas 1-3 to the more general and more elegant C^*-algebra approach to scattering by Araki & Haag (1967).

Appendix 1: *Proof of Lemma* 1

Without loss of generality we assume φ_0 to be non-trivial. Then by†

$$a_r \equiv \max_{\substack{r', r'' \in Z_+ \\ r', r'' \leqslant r}} \max_{\substack{a, b \in Z_+^n \\ |a|, |b| \leqslant r}} \max_{t \in R^n} |t^a D_t^b (\|(t^1, \ldots, t^{n'})\|^{4r'} \varphi_{r''}(t))|$$

with $r \in Z_+$ ($\equiv Z_+^1$) we define a non-decreasing sequence $\{a_r\}$ of positive numbers. Hence

$$\psi(t') \equiv \sum_{r=0}^{\infty} \frac{\|t'\|^{4r}}{(2r)!a_r}; \qquad t' \in R^{n'}$$

is a positive entire function. Although $\psi(t')$ cannot be a multiplier in $\mathscr{S}(R^{n'})$, it is quite evident that the functions h_r defined by

$$h_r(t) \equiv \psi(t^1, \ldots, t^{n'})\varphi_r(t^1, \ldots, t^n); \qquad t \in R^n, r \in Z_+$$

are all tempered (i.e. $h_r \in \mathscr{S}(R^n)$). On the other hand, the function

$$\psi_1(t^1) \equiv \sum_{r=0}^{\infty} \frac{(t^1)^{2r}}{(2r)!a_r}; \qquad t^1 \in R^1$$

fulfils the inequalities

$$|\psi_1^{(r)}(t^1)| \leqslant (1 + |t^1|)\psi_1(t^1); \qquad r \in Z_+$$

Therefore, the identity

$$(1/\psi_1)^{(r+1)} = (\psi_1^{(r+1)} - \sum_{j=1}^{r+1} \binom{r+1}{j} (\psi_1^2)^{(j)} (1/\psi_1)^{(r+1-j)})/\psi_1^2; \qquad r \in Z_+$$

shows that $1/\psi_1 \in S(R^1)$. Hence, for $g \equiv 1/\psi$ we get all the required properties:

$$g > 0, \qquad g \in \mathscr{S}(R^{n'}), \qquad \varphi_r(t) = g(t^1, \ldots, t^{n'})h_r(t)$$

Appendix 2: *Proof of Statement* 1 *of Lemma* 2

We just have to give an explicit version of Ruelle's sketchy proof (Ruelle, 1962) (see also Araki (1962) for the simple case $1 \leqslant v \equiv (\mathbf{v}^2)^{1/2}$).

Let us choose some infinitely differentiable function g over R^1 with

$$g(t) = \begin{cases} 1 & \text{for } |t| < 1 \\ 2 & \text{for } |t| > 2. \end{cases}$$

† See footnote on page 94 for definitions.

Then the $\mathcal{O}_{\mathcal{M}}(R^3)$-functions†

$$k_1(\mathbf{p}) \equiv \theta(v-1) + \theta(1-v)g(p^2)g(p^3)$$
$$k_2(\mathbf{p}) \equiv \theta(1-v)(1-g(p^2))$$
$$k_3(\mathbf{p}) \equiv \theta(1-v)g(p^2)(1-g(p^3))$$

have the following properties:

$$k_1(\mathbf{p}) + k_2(\mathbf{p}) + k_3(\mathbf{p}) = 1 \tag{A.2.1}$$
$$|p^2|, |p^3| < 2 \qquad \text{if } \theta(1-v)k_1(\mathbf{p}) \neq 0$$
$$|p^2| > 1 \qquad \text{if } k_2(\mathbf{p}) \neq 0$$
$$|p^3| > 1 \qquad \text{if } k_3(\mathbf{p}) \neq 0$$

Now, if we define

$$\mathbf{q} \equiv (mv/(1-v^2)^{1/2}, 0, 0) \qquad \text{for } v < 1$$
$$a \equiv \left(1 + 9\theta(1-v)\frac{m^4 + 16\|\mathbf{p}\|^2}{m^4\|\mathbf{p}-\mathbf{q}\|^2}\right)^{-1} \qquad \text{for } \mathbf{p} \neq \mathbf{q}$$

we have the inequalities

$$(1/2)\min\left\{1, \frac{m^4\|\mathbf{p}-\mathbf{q}\|^2}{9(m^4 + 16\|\mathbf{p}\|^2)}\right\} \leqslant a < \min\left\{(1/3)\|\mathbf{p}-\mathbf{q}\|, (m/2)\sqrt{\frac{\|\mathbf{p}-\mathbf{q}\|}{3\|\mathbf{p}\|}}\right\}$$
$$\|\mathbf{p}-\mathbf{q}\| < 3|p^1 - q^1| \qquad \text{if } |p^2|, |p^3| < 2a$$

for $v < 1$. Hence the rough estimate

$$w_{\mathbf{p}}\left|\frac{\partial}{\partial p^j}(w_{\mathbf{p}}u^0 - \|\mathbf{u}\|p^1)\right|$$
$$= \theta(v-1)|p^j u^0 - \|\mathbf{u}\|w_{\mathbf{p}}\delta^j_1| + \theta(1-v)\frac{|p^j w_{\mathbf{q}} - \|\mathbf{q}\|w_{\mathbf{p}}\delta^j_1|}{(m^2 + 2\|\mathbf{q}\|^2)^{1/2}}$$
$$> m^4/(36(2 + m^4 + 16\|\mathbf{p}\|^2)^3(1 + \theta(1-v)\|\mathbf{p}-\mathbf{q}\|^{-2}))^{\cdot} > 0$$
$$\text{for } \mathbf{p} \in \operatorname{supp} k_j(\mathbf{p}/a)\hat{f}(\mathbf{R}^{-1}\mathbf{p}) \tag{A.2.2}$$

is easily established for every spatial rotation $\mathbf{R}$ with

$$\mathbf{R}\mathbf{u} = (\|\mathbf{u}\|, 0, 0)$$

where

$$u^0 \equiv 1/(1+v^2)^{1/2}, \qquad \mathbf{u} \equiv \mathbf{v}/(1+v^2)^{1/2}$$

† Without loss of generality we assume $v \neq 1$ throughout the proof. As usual, the characteristic function of the positive real axis is denoted by θ.

By (A.2.1) we have

$$(2\pi)^{3/2} f_x(t, \mathbf{v}t) = \sum_{j=1}^{3} \int \frac{d\mathbf{p}}{2w_\mathbf{p}} (k_j(\mathbf{p}/a) h(\mathbf{R}^{-1}\mathbf{p}) \hat{f}(\mathbf{R}^{-1}\mathbf{p}))$$
$$\exp[-il(w_\mathbf{p} u^0 - \|\mathbf{u}\| p^1)]$$

with

$$f_x(t, \mathbf{v}t) \equiv f(t - x^0, \mathbf{v}t - \mathbf{x}), \qquad h_x(\mathbf{p}) \equiv \exp[i(w_\mathbf{p} x^0 - \mathbf{p}\mathbf{x})],$$
$$l \equiv \|(t, \mathbf{v}t)\| \tag{A.2.3}$$

By (A.2.2) we are allowed to perform the substitution

$$p^j \to \xi \equiv w_\mathbf{p} u^0 - \|\mathbf{u}\| p^1$$

in the integral

$$\int \frac{dp_j}{2w_\mathbf{p}} k_j(\mathbf{p}/a) h_x(\mathbf{R}^{-1}\mathbf{p}) \hat{f}(\mathbf{R}^{-1}\mathbf{p}) \exp[-il\xi(p^j)]$$
$$= \int d\xi \left(w_\mathbf{p} \frac{\partial \xi}{\partial p^j} \right)^{-1} k_j(\mathbf{p}/a) h_x(\mathbf{R}^{-1}\mathbf{p}) \hat{f}(\mathbf{R}^{-1}\mathbf{p})\, e^{il\xi}$$

(of course, we have $p^j = p^j(\xi)$ on the right-hand side). Thus, by successive N-fold partial integration with respect to ξ and resubstitution $\xi \to p^j$ we obtain:

$$(2\pi)^{3/2} l^N |f_x(t, \mathbf{v}t)|$$
$$\leqslant \sum_{j=1}^{3} \int d\mathbf{p} \left| \frac{\partial \xi}{\partial p^j} \left(\left(\frac{\partial \xi}{\partial p^j} \right)^{-1} \frac{\partial}{\partial p^j} \right)^n \left(\left(w_\mathbf{p} \frac{\partial \xi}{\partial p^j} \right)^{-1} k_j(\mathbf{p}/a) h_x(\mathbf{R}^{-1}\mathbf{p}) \hat{f}(\mathbf{R}^{-1}\mathbf{p}) \right) \right|$$

By inspection of

$$\left(\frac{\partial}{\partial p^j} \right)^r k_j(\mathbf{p}/a) \qquad \text{and} \qquad \left(\frac{\partial}{\partial p^j} \right)^r \left(\frac{\partial \xi}{\partial p^j} \right)^{-1},$$

the latter in connection with (A.2.2), we see that

$$\|(t, \mathbf{v}t)\|^N |f_x(t, \mathbf{v}t)|$$
$$\leqslant A' \max_{|\mathbf{a}|, |\mathbf{b}| \leqslant N} \int d\mathbf{p}\, |w_\mathbf{p}^r (D_\mathbf{p}^\mathbf{a} h_x(\mathbf{R}^{-1}\mathbf{p}))(1 + \theta(1 - v)\|\mathbf{p} - \mathbf{q}\|^{-r'}) D_\mathbf{p}^\mathbf{b} f(\mathbf{R}^{-1}\mathbf{p})|$$

holds with suitable integers A', r, r' which can be chosen independent of x, $\hat{f}$, $\mathbf{v}$, and $\mathbf{R}$. Hence, by the mean value theorem (for differentiation), we have (remember that $\hat{f}(\mathbf{R}^{-1}\mathbf{q}) = 0$ if $v < 1$):

$$\|(t, \mathbf{v}t)\|^N |f_x(t, \mathbf{v}t)|$$
$$\leqslant A'' \max_{|\mathbf{a}| \leqslant N} \max_{|\mathbf{b}| \leqslant N + r'} \int d\mathbf{p}\, |w_\mathbf{p}^r (D_\mathbf{p}^\mathbf{a} h_x(\mathbf{p}))| \sup_{\substack{\mathbf{p}' \in R^3 \\ \|\mathbf{p} - \mathbf{p}'\| < 1}} |D_{\mathbf{p}'}^\mathbf{b} \hat{f}(\mathbf{p}')|$$

According to the definition of f_x and h_x in (A.2.3), statement 1 of Lemma 2 is just a simple consequence of this inequality.

Appendix 3: Non-existence of Hepp's Partition of the Unity

Let us restrict $\mathbf{q} \in R^3$ to the positive sector S_{12}^+ of the $q^1 - q^2$-plane. Then the curves of constant $q^1/w_\mathbf{q}$ are characterised by $dq^2/dq^1 = q^1 q^2/[m^2 + (q^2)^2]$ and the curves of constant $q^2/w_\mathbf{q}$ by $dq^2/dq^1 = [m^2 + (q^1)^2]/(q^1 q^2)$. Therefore, we can choose a sequence $\{\mathbf{q}_r\} \subset S_{12}^+$ with the following properties ($r = 1, 2, 3, \ldots$):

$$|r - q_{3r-1}^l| < 1/2 \qquad \text{for } l \in \{1, 2\},\ j \in \{0, 1, 2\}$$
$$q_{3r-1}^2| > q_{3r-1}^1, \qquad q_{3r-j}^2 < q_{3r-j}^1 \qquad \text{for } j = 1, 2$$
$$q_{3r}^1/w_{\mathbf{q}_{3r}} = q_{3r-1}^1/w_{\mathbf{q}_{3r-1}}, \qquad q_{3r-1}^2/w_{\mathbf{q}_{3r-1}} = q_{3r-2}^2/w_{\mathbf{q}_{3r-2}} \tag{A.3.1}$$

$$\|\mathbf{q}_{3r} - \mathbf{q}_{3r-2}\| < \exp(-r) \tag{A.3.2}$$

Furthermore, we may choose† functions $f_1, f_2 \in \mathscr{S}(R^3)$ with the properties:

$$f_1(\mathbf{q}_{3r-1}) \neq 0 \qquad \text{for } r = 1, 2, \ldots \tag{A.3.3}$$
$$|q^2| > |q^1| \qquad \text{if } \mathbf{q} \in \operatorname{supp} f_1 \tag{A.3.4}$$
$$f_2(\mathbf{q}_{3r}) \neq 0 \neq f_2(\mathbf{q}_{3r-2}) \qquad \text{for } r = 1, 2, \ldots \tag{A.3.5}$$
$$|q^2| < |q^1| \qquad \text{if } \mathbf{q} \in \operatorname{supp} f_2 \tag{A.3.6}$$

The essential features of the whole construction are illustrated by Fig. 1.

Now, let us consider the special case

$$k = 3, \qquad \chi(\mathbf{p}_2, \mathbf{p}_3) = f_1(\mathbf{p}_2) f_2(-\mathbf{p}_2 - \mathbf{p}_3)$$

Since by (A.3.4) and (A.3.6) we have $\mathbf{p}_2 + \mathbf{p}_3 \neq \mathbf{p}_2$ in supp χ, the condition

$$\left[m^2 + \left(\sum_{j=2}^{k} \mathbf{p}_j\right)^2\right]^{-1/2} \sum_{j=2}^{k} \mathbf{p}_j + (m^2 + \mathbf{p}_2^2)^{-1/2}\mathbf{p}_2 \neq 0 \qquad \text{in supp } \chi(\mathbf{p}_2, \mathbf{p}_3)$$

is evidently fulfilled. Let $\alpha_1, \alpha_2, \alpha_3$ be arbitrarily differentiable functions over R^6 with

$$\alpha_1 + \alpha_2 + \alpha_3 = 1 \tag{A.3.7}$$

and

$$\left[m^2 + \left(\sum_{j=2}^{k} \mathbf{p}_j\right)^2\right]^{-1/2} \sum_{j=2}^{k} p_j^l + (m^2 + \mathbf{p}_2^2)^{-1/2} p_2^l \neq 0 \qquad \text{in supp } \chi\alpha_l \tag{A.3.8}$$

† Those functions can be easily constructed by standard techniques.

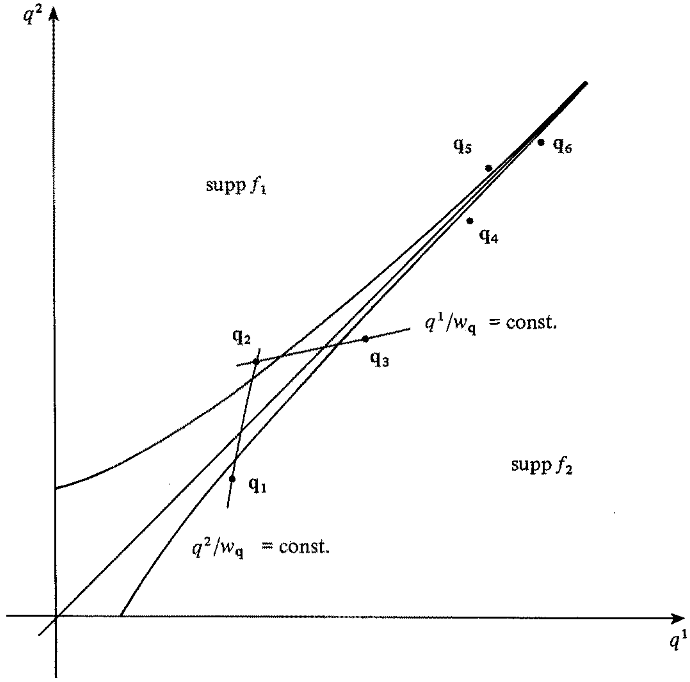

Figure 1.

for $l = 1, 2, 3$. Then, in order to disprove Hepp's assumption (see Introduction), we only have to show that $\alpha_1 \notin \mathcal{O}_{\mathcal{M}}(R^6)$:

Since by (A.3.3) and (A.3.5) we have

$$\chi(\mathsf{q}_{3r-1}, -\mathsf{q}_{3r-1} - \mathsf{q}_{3r-j}) \neq 0 \qquad \text{for } j = 0, 2 \text{ and } r = 1, 2, \ldots \tag{A.3.9}$$

(A.3.8) can hold for $l = 3$ only if:

$$\alpha_3(\mathsf{q}_{3r-1}, -\mathsf{q}_{3r-1} - \mathsf{q}_{3r-j}) = 0 \qquad \text{for } j = 0, 2 \text{ and } r = 1, 2, \ldots \tag{A.3.10}$$

Similarly, by (A.3.1) and (A.3.9), (A.3.8) can hold for $l = 1, 2$ only if:

$$\alpha_1(\mathsf{q}_{3r-1}, -\mathsf{q}_{3r-1} - \mathsf{q}_{3r}) = \alpha_2(\mathsf{q}_{3r-1}, -\mathsf{q}_{3r-1} - \mathsf{q}_{3r-2}) = 0 \tag{A.3.11}$$

(A.3.7), (A.3.10), and (A.3.11) imply:

$$\alpha_1(\mathsf{q}_{3r-1}, -\mathsf{q}_{3r-1} - \mathsf{q}_{3r-2}) = 1 \tag{A.3.12}$$

Finally, the mean value theorem (for differentiation), (A.3.2), (A.3.11), and (A.3.12) show that at least one of the functions

$$\frac{\partial}{\partial p_3^1}\alpha_1(\mathbf{p}_2, \mathbf{p}_3), \qquad \frac{\partial}{\partial p_3^2}\alpha_1(\mathbf{p}_2, \mathbf{p}_3)$$

cannot be polynomially bounded. Thus $\alpha_1 \notin \mathcal{O}_{\mathcal{M}}(R^6)$, indeed.

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